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Analytic solutions for cosmological perturbations in multi-dimensional space-time ^{*} [†]

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Abstract

We obtain analytic solutions for the density contrast and the anisotropic pressure in a multi-dimensional FRW cosmology with collisionless, massless matter. These are compared with perturbations of a perfect fluid universe. To describe the metric perturbations we use manifest gauge invariant metric potentials. The matter perturbations are calculated by means of (automatically gauge invariant) finite temperature field theory, instead of kinetic theory.

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The study of multi-dimensional cosmological models [1] may give some hints to explain the existence of just four visible space-time dimensions. Contracting internal dimensions (Kaluza-Klein cosmologies) could produce observable effects in our visible space-time, e.g. gravity waves [2] or a de Sitter phase (producing “enough” inflation in $D \gtrsim 40$ [3]).

Cosmological perturbations [4] are believed to provide the seeds for the large scale structure in our universe. The task of this work is to find out how the evolution of cosmological perturbations depends on the dimension of the universe.

We consider a radiation dominated, spatially flat Friedman-Robertson-Walker (FRW) background in arbitrary dimension $D \geq 4$. This background is chosen for reasons of simplicity. Cosmological perturbations for collisionless, massless matter are calculated and compared with perfect fluid perturbations. In contrast to this work, Fabris and Martin [5] dealt with perturbations of Kaluza-Klein models without matter in their contribution to this year’s Journées Relativistes. Our units are fixed by $\hbar = c = k_B = 1$.

Cosmological perturbations can be classified into scalar, vector and tensor perturbations[§]. An appropriate tool for describing the corresponding metric perturbations $\delta g_{\mu\nu}(x)$ is provided by Bardeen’s [6] gauge invariant metric potentials[¶]. In the following, we only deal with scalar perturbations. Their two metric potentials will be denoted by Φ and Π (their definition differs from that of Bardeen’s ϕ_A and ϕ_H). They are related to the density contrast δ and the anisotropic pressure π_{anis} , i.e.

$$\delta(x) = \frac{(D-2)^2}{4} \frac{x^2}{D-1} \Phi(x) \quad (1)$$

$$\pi_{anis}(x) = \frac{D-2}{2} \frac{x^2}{D-1} \Pi(x), \quad (2)$$

on a space-like hypersurface representing the local restframe of matter everywhere. Here $x = k\tau$, where k denotes the wavenumber of the perturbations and τ the conformal time of the FRW background.

The evolution of Φ and Π is determined by the Einstein-Jacobi equations

$$\delta(\sqrt{-g}G_{\mu\nu}) = 8\pi G \delta(\sqrt{-g}T_{\mu\nu}),$$

or equivalently

$$\delta(\sqrt{-g}G_{\mu\nu}) = 16\pi G \delta\left(\frac{\delta\Gamma^M}{\delta g^{\mu\nu}}\right). \quad (3)$$

$G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ the energy momentum tensor. Γ^M denotes the matter part of the effective action.

The calculation of the r.h.s. of Eq. (3) is done by means of finite temperature field theory. This method was developed by Kraemmer and Rebhan [8]. The second

[§]This classification is due to space-coordinate transformations on the constant time hypersurface.

[¶]For a recent review see Ref. [7].

variation of the matter effective action (the r.h.s. of Eq. (3)) is the contribution of gravitationally coupled matter to the graviton self-energy. Here we restrict ourselves to the case of *collisionless*, massless matter, e.g. background gravitons and neutrinos after their decoupling. To calculate perturbatively the graviton self-energy

$$\frac{\delta^2 \Gamma^M}{\delta g^{\alpha\beta}(x) \delta g^{\mu\nu}(y)} =: \sqrt{-g} \Pi_{\alpha\beta\mu\nu}(x, y) \quad (4)$$

we keep the temperature T below the Planck scale, i.e. $T^{D-2} \ll G^{-1}$. From the Einstein equation and the radiation energy-momentum tensor $T_{\mu\nu} \sim T^D$ we estimate $R_{\mu\nu} \sim GT^D \ll T^2$. Thus curvature terms are of subleading order. On the other hand, we assume that the temperature is much higher than the incoming momenta (high temperature expansion). So only high temperature 1-loop contributions ($\sim T^D$) to (4) have to be calculated. This has been done in Ref. [9] for $D = 4$. The advantage of the field theory approach — instead of solving the Boltzmann equation — is the guaranteed gauge invariance.

Having calculated $\Pi_{\alpha\beta\mu\nu}$ we have to solve the Einstein-Jacobi equations (3) for Φ and Π . They read:

$$\Phi''(x) + \frac{4}{x} \Phi'(x) + \frac{1}{D-1} \Phi(x) = \frac{2}{D-1} \Pi(x) - \frac{4}{D-2} \frac{1}{x} \Pi'(x) \quad (5)$$

$$\begin{aligned} & \left(\frac{D-2}{2} x^2 + D-1 \right) \Phi(x) + (D-1) x \Phi'(x) = \\ & 2(D-1) \left(\frac{D-2}{2} \Phi + \Pi \right) (x) - \frac{4(D-1)D}{\sqrt{\pi}(D-2)} \Gamma \left(\frac{D-1}{2} \right) \times \\ & \left[\int_0^x \left(\frac{2}{x-x'} \right)^{\frac{D-4}{2}} \sqrt{\frac{\pi}{2(x-x')}} J_{\frac{D-3}{2}}(x-x') \left(\frac{D-2}{2} \Phi + \Pi \right)'(x') dx' + \right. \\ & \left. + \left(\frac{2}{x} \right)^{\frac{D-4}{2}} \sqrt{\frac{\pi}{2x}} \sum_{n=0}^{\infty} \gamma_n J_{\frac{D-3+2n}{2}}(x) \right] \quad (6) \end{aligned}$$

Eq. (5) is the trace of Eq. (3), whereas Eq. (6) corresponds to its 0-0 component. The initial conditions are implemented by fixing the arbitrary constants γ_n in Eq. (6). We simply set $\gamma_0 = \text{const}$ and $\gamma_n = 0$ for $n \geq 1$. This choice corresponds to the initial values: $\frac{\Pi(0)}{\Phi(0)} = -\frac{D-3}{D+3}$. In kinetic theory similar equations to (5) and (6) have been derived in particular gauges and studied in Refs. [10]. Solutions have been obtained numerically only.

Exact analytic solutions to (5) and (6) may be found by a power series Ansatz [8] yielding for the first terms:

$$\begin{aligned} \Phi(x) &= \text{const} \left((D+3) - \frac{3D^3 + 11D^2 - 20D - 30}{5D^2 - 4D - 10} \frac{x^2}{2!} + \dots \right) \\ \Pi(x) &= \text{const} \left(-(D-3) + \frac{(D-2)(11D^2 - 17D - 30)}{2(D+3)(5D^2 - 4D - 10)} \frac{x^2}{2!} - \dots \right). \end{aligned}$$

The metric potentials can be calculated with arbitrary precision since the series are rapidly converging for all x . This determines the density contrast and the anisotropic pressure through Eqs. (1) and (2). The results are plotted in Fig. 1 for $D = 4$, in Fig. 2 for $D = 5$, and in Fig. 3 for $D = 8$.

Density perturbations of *collisional*, massless matter, e.g. photons before recombination, are also plotted in Figs. 1 – 3. They are given by the solution of Eq. (5) with Π vanishing. This happens, because the photons interact via Thomson scattering, and therefore no anisotropic pressure can evolve. Matter remains a perfect fluid. The solution $\delta_{pf}(x) \sim x j_1(\frac{x}{\sqrt{D-1}})$ has already been obtained in Ref. [11].

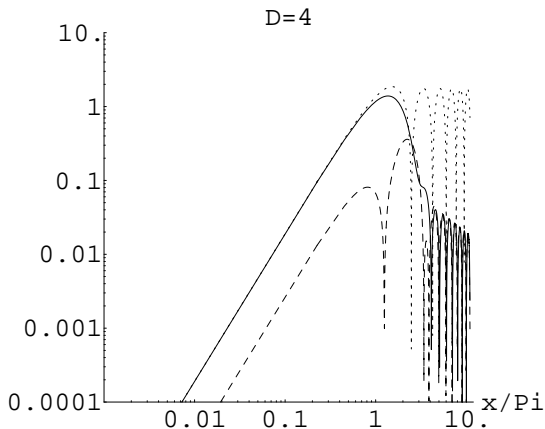


Figure 1: The absolute values of the density contrast $|\delta|$ (full line) and the anisotropic pressure $|\pi_{anis}|$ are plotted over x/π for a four dimensional FRW universe, matter being massless and collisionless. The density contrast for collisional matter $|\delta_{pf}|$ is shown by the dotted line.

To discuss Figs. 1 – 3 note that $\frac{x}{\pi} = (\frac{\lambda}{2})^{-1} R_H$ (R_H being the comoving horizon size). Far outside the horizon ($\frac{x}{\pi} \ll 1$) all perturbations grow as x^2 — independent of dimension. This growth is related to the gravitational (Jeans) instability.

Well inside the horizon ($\frac{x}{\pi} \gg 1$) the perturbations oscillate. For perfect fluids their amplitude is constant. Their wavelength is determined by their dimension dependent sound velocity ($v_S = \frac{1}{\sqrt{D-1}}$). For collisionless matter the perturbations are damped ($\sim x^{-(D-2)/2}$) due to directional dispersion. This dimension dependent effect can be understood easily [12]. In more than one dimension peaks and troughs of waves travel in different directions, and consequently the wave is damped. These waves are propagating with the speed of light.

Another effect is observed in Figs. 1 – 3. The maximum of the perturbations is moving to higher values of x with higher dimension. The maximum value itself is

fixed by the arbitrary constant γ_0 .

We showed that the method developed in Refs. [9, 8, 13] works in multi-dimensional space-time, too. Vector^{||} and tensor perturbations could be calculated in the same manner. Two dimension dependent effects have been demonstrated: Damping of collisionless perturbations is due to directional dispersion, and perturbations start to “feel” the horizon at a later time in higher dimensional universes.

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^{||}A detailed treatment of vector perturbations for $D = 4$ is given in Ref. [13].

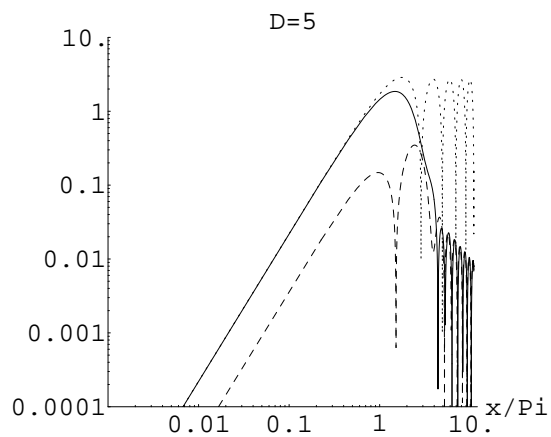


Figure 2: As Fig. 1, but $D = 5$.

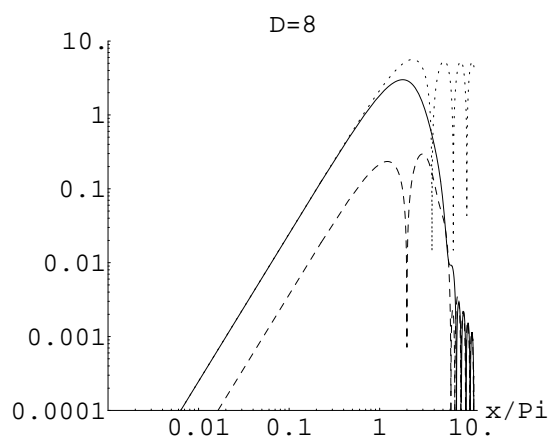


Figure 3: As Fig. 1, but $D = 8$.